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# RELATIONS AMONG SPLITTINGS OF COHOMOLOGIES OF $p$ -GROUPS WITH RANK 2 (Cohomology theory of finite groups and related topics)

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## RELATIONS AMONG SPLITTINGS OF COHOMOLOGIES OF $p$ -GROUPS WITH RANK 2

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### 1. INTRODUCTION

Let  $P$  be a  $p$ -group and  $BP$  its classifying space. We study the stable splitting and splitting of cohomology

$$(*) \quad BP \cong X_1 \vee \dots \vee X_i,$$

$$(**) \quad H^*(P) \cong H^*(X_1) \oplus \dots \oplus H^*(X_i) \quad (for \ * > 0)$$

where  $X_i$  are irreducible spaces in the stable homotopy category.

From the answer of the Segal conjecture by Carlsson, the splittings  $(*)$  are given by only using modular representation theory by Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. In fact, their theorems say that such a decomposition is decided only by structures of simple modules of the  $\text{mod}(p)$  double Burnside algebra  $A(P, P)$ . These theorems do not use splittings of cohomology  $(**)$ .

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting  $(*)$  for groups  $P$  with  $\text{rank}_p(P) = 2$  for  $p \geq 5$ . However it was not used splittings  $(**)$  of the cohomology  $H^*(P)$ , and the cohomologies  $H^*(X_i)$  were not given there. In [Hi-Ya1], [Hi-Ya2], we give the cohomology  $H^*(X_i)$  (and hence  $(**)$ ) for  $P = p_+^{1+2}$  the extraspecial  $p$  group of order  $p^3$  and exponent  $p$ . Their cohomology  $H^*(X_i)$  are very complicated but have rich structures, in fact  $p_+^{1+2}$  is a  $p$ -Sylow subgroup of many interesting groups, e.g.,  $GL_3(\mathbb{F}_p)$  and many simple groups e.g.  $J_4$  for  $p = 3$ .

In this paper, we give the decomposition of  $H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0})$  for other  $\text{rank}_p P = 2$  groups for odd primes  $p$ . In fact the double Burnside algebra  $A(P, P)$  acts on  $\sqrt{0} \subset H^*(P; \mathbb{Z})$ .

In most cases,  $H^*(X_i)$  are seemed not to have so rich structure as  $p_+^{1+2}$ , since  $H^*(P) \subset H^*(p_+^{1+2})$  as graded modules, and they are not

$p$ -Sylow subgroups of so interesting groups. However, we hope that from our computations, it becomes more clear that the relations among splittings of  $H^*(P)$  of groups  $P$  with  $\text{rank}_p(P) = 2$ . In particular, we note that the irreducible components of  $Bp_+^{1+2}$  are most *fine* in those of  $\text{rank}_p = 2$  groups, namely, the cohomology  $H^*(X_i(P))$  can be written as a sum of submodules of  $H^*(X_k(p_+^{1+2}))$  ( Theorem 7.2, Theorem 7.5, Corollary 7.6).

**Theorem 1.1.** *For  $p \geq 5$ , let  $P$  be a non-abelian  $p$ -group of  $\text{rank}_p P = 2$ , which is not a metacyclic group. For each primitive idempotent  $e \in A(P, P)$ , there is an idempotent  $f \in A(p_+^{1+2}, p_+^{1+2})$  such that  $eH^*(P) \cong fH^*(p_+^{1+2})$ . Namely, for each irreducible component  $X_i(P)$  of  $BP$ , we can take some index set  $J(i, P)$  such that*

$$H^*(X_i(P)) \cong \bigoplus_{j \in J(i, P)} H^*(X_j(p_+^{1+2})).$$

**Remark.** Note that  $X_i(P) \not\cong \bigvee_{j \in J(i, P)} X_j(p_+^{1+2})$  in the stable homotopy category, because  $X_i(P)$  is irreducible.

This paper is planed as follows. In §2 we recall the relation between  $A(P, P)$  and the stable splitting. In §3, we note  $\text{Out}(P)$ -actions. In §4 – §6, we give the decomposition of  $H^*(P)$  for metacyclic groups,  $C(r)$  groups (such that  $C(3) = p_+^{1+2}$ ),  $G(r, e)$  groups respectively. In §7, we study the relation of splittings among groups studied in §4 – §6.

## 2. THE DOUBLE BURNSIDE ALGEBRA AND STABLE SPLITTING

Let us fix an odd prime  $p$  and  $k = \mathbb{F}_p$ . For a finite groups  $G$ , let  $A_{\mathbb{Z}}(G, G)$  be the double Burnside group defined by the Grothendieck group generated by  $(G, G)$ -bisets with free right  $G$ -actions. Each element  $\Phi$  in  $A_{\mathbb{Z}}(G, G)$  is generated by elements  $[Q, \phi] = (G \times_{(Q, \phi)} G)$  for some subgroup  $Q \leq G$  and a homomorphism  $\phi : Q \rightarrow G$ . In this paper, we use the notation

$$[Q, \phi] = \Phi : G \geq Q \xrightarrow{\phi} G.$$

By the composition, the group  $A_{\mathbb{Z}}(G, G)$  becomes a ring, and call it the (integral) double Burnside algebra.

For each  $\Phi = [Q, \phi] \in A_{\mathbb{Z}}(G, G)$ , we can define a  $\Phi$ -action on  $H^*(G; k)$  by

$$\Phi(x) = [Q, \phi] \cdot x = \text{Tr}_Q^G \phi^*(x) \quad \text{for } x \in H^*(G; k).$$

In particular, for a finite group  $G$ , we have an  $A_{\mathbb{Z}}(G, G)$ -module structure on  $H^*(G; k)$  and  $H^*(G; \mathbb{Z})$ .

Recall Quillen's theorem [Qu] such that the restriction map  $H^*(G; k) \rightarrow \lim_V H^*(V; k)$  is an F-isomorphism (i.e. the kernel and cokernel are

nilpotent) where  $V$  ranges elementary abelian  $p$ -subgroups of  $G$ . Using this theorem, it is easily seen ([Hi-Ya1])

**Lemma 2.1.** *Let  $\sqrt{0}$  be the nilpotent ideal in  $H^*(G; k)$  (or  $H^*(G; \mathbb{Z})/p$ ). Then  $\sqrt{0}$  itself is an  $A_{\mathbb{Z}}(G, G)$ -module.*

In this paper we consider the cohomology modulo nilpotent elements. We simply write

$$H^*(G) = H^*(G; \mathbb{Z})/(p, \sqrt{0}).$$

By the preceding lemma,  $H^*(G)$  has the  $A_{\mathbb{Z}}(G, G)$ -module structures.

Let a ring  $R$  act on  $H^*(G)$  (e.g.,  $R = A_{\mathbb{Z}}(G, G)$ ,  $k[Out(G)]$ ). Suppose that there is an  $R$ -filtration  $F_1 \subset \dots \subset F_n \cong H^*(P)$  such that

$$gr H^*(G) = \bigoplus F_{i+1}/F_i \cong \bigoplus m_j M_j \quad \text{for } * > 0$$

with simple  $R$ -modules  $M_j$ . Then we write  $H^*(G) \leftrightarrow \bigoplus m_j M_j$ .

Throughout this paper, we assume that degree  $*$   $> 0$  so that  $H^*(X \vee X') \cong H^*(X) \oplus H^*(X')$ . In this paper,  $H^*(G) \cong A$  for an graded ring  $A$  means an graded module isomorphism otherwise stated, while (in most cases) it is induced from the ring isomorphism  $gr H^*(G) \cong A$  for some filtrations of  $H^*(G)$ .

Let  $BG = BG_p$  be the  $p$ -completion of the classifying space of  $G$ . Recall that  $\{BG, BG\}_p$  is the ( $p$ -completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem [Ca], [Ma-Pr]) that this group is isomorphic to the double Burnside algebra  $A_{\mathbb{Z}}(G, G)^\wedge$  completed by the augmentation ideal. Since the transfer is represented as a stable homotopy map  $Tr$ , an element  $\Phi = [Q, \phi] \in A(G, G)$  is represented as a sum of maps  $\Phi \in \{BG, BG\}_p$

$$\Phi : BG \xrightarrow{Tr} BQ \xrightarrow{B\phi} BG.$$

Of course, for  $x \in H^*(G)$ , we have  $Tr^*(B\phi)^*(x) = Tr_Q^G \phi^*(x)$ .

Let us write  $A(G, G) = A_{\mathbb{Z}}(G, G) \otimes k$ . Hereafter we consider in the case  $G = P$  for a  $p$ -group  $P$ . Given a primitive idempotents decomposition of the unity of  $A(P, P)$

$$1 = e_1 + \dots + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee \dots \vee X_n \quad \text{with } e_i BP = X_i.$$

In this paper, an isomorphism  $X \cong Y$  for spaces means that it is a stable homotopy equivalence. Recall that  $M_i = A(P, P)e_i/(rad(A(P, P)e_i)$  is

a simple  $A(P, P)$ -module where  $\text{rad}(-)$  is the Jacobson radical. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \bigvee_j \left( \bigvee_k X_{jk} \right) = \bigvee_j m_j X_{j1} \quad \text{where } m_j = \dim(M_j)$$

where  $A(P, P)e_{jk}/\text{rad}(A(P, P)e_{jk}) \cong M_j$  for all  $k$ , and  $\dim(M_j)$  is the dimension of  $M_j$  over the field  $\text{End}(M_j)$  which is isomorphic to  $k$  in cases considered in this paper. Therefore the stable splitting of  $BP$  is completely determined by the idempotent decomposition of the unity in the double Burnside algebra  $A(P, P)$ .

Here  $X_i$  is only defined in the stable homotopy category. (So strictly, the cohomology ring  $H^*(X_i)$  is not defined.) However we can define  $H^*(X_i)$  as a graded submodule of the cohomology ring  $H^*(P)$  by

$$H^*(X_i) = e_i \cdot H^*(P) \quad (= e_i^* H^*(P) \text{ stably}).$$

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple  $A(P, P)$ -module is written as

$$S(P, Q, V) \quad \text{for } Q \leq P, \text{ and } V : \text{simple } k[\text{Out}(Q)] - \text{module}.$$

Then the main theorem of stable splitting of  $BP$  is stated as follow.

**Theorem 2.2.** (Benson-Feshbach [Be-Fe], Martino-Priddy [Ma-Pr]) *There are indecomposable stable spaces  $X_{S(P, Q, V)}$  for  $S(P, Q, V) \neq 0$  such that*

$$BP \cong \bigvee_{Q \leq P} (\dim S(P, Q, V)) X_{S(P, Q, V)}.$$

### 3. $\text{Out}(P)$ -MODULES

Let  $R$  be a subring of  $A(P, P)$ . For a simple  $R$ -module  $S_R$ , we can define the idempotent  $e_{S_R}$  and the stable space  $Y_{S_R} = e_{S_R} BP$  which decomposes  $BP$ , while it is (in general) not irreducible. In particular, we take the group algebra  $k[\text{Out}(P)]$  of the outer automorphism group  $\text{Out}(P)$  as the ring  $R$ .

**Lemma 3.1.** *For  $\text{Out}(P)$ -simple modules  $R_i$  with  $\dim(R_i) = n_i$ , we have*

$$BP = n_1 Y_1 \vee \dots \vee n_s Y_s \quad \text{where } Y_i = e_{R_i} BP$$

for idempotents  $e_{R_i}$  in  $k[\text{Out}(P)]$ . Then each  $Y_i$  decomposes

$$Y_i = m_{i,1} X_{S_{i1}} \vee \dots \vee m_{i,t} X_{S_{it}} \quad \text{for } X_{S_{ij}} = e_{S_{ij}} BP$$

where  $e_{S_{ij}}$  are idempotents in  $A(P, P)$  with  $\dim(S_{ij}) = n_i m_{i,j}$ .

An irreducible summands  $X_{S(P,Q,V)}$  are called dominant summands if  $Q = P$  ([Ni], [Ma-Pr]). Let  $X_{S=S(P,Q,V)}$  be a non-dominant summand for a proper subgroup  $Q$ . Then it is known ([Ni],[Ma-Pr]) that the corresponding idempotent  $e_S \in A(P, P)$  is generated by elements  $P > Q \rightarrow P$  and  $P \rightarrow Q \rightarrow P$ . Hence when there is no non-trivial map  $P \rightarrow Q$ , we see  $H^*(X_S) \cong e_S H^*(BP) \subset Tr_Q^P H^*(Q)$ .

**Corollary 3.2.** *Let  $V$  be a simple  $Out(P)$ -module. Then we have decomposition*

$$Y_V \cong X_{S(P,P,V)} \vee \bigvee_{Q \neq P} X_{S(P,Q,W)}.$$

For a simple  $Out(P)$ -module  $V$ , define a stable summand  $Y(V)$  by

$$e_V = \sum_{V_i \cong V} e_i, \quad Y(V) = \bigvee_{V_{jk} \cong V} Y_{jk} = e_V BP.$$

**Lemma 3.3.** *Given a simple  $Out(P)$ -module  $V$ , we have*

$$H^*(Y(V)) \leftrightarrow \bigoplus_{i=1} V[k_i], \quad 0 \leq k_1 \leq \dots \leq k_s \leq \dots$$

where  $[k_s]$  is the operation ascending degree  $k_s$ .

In this paper, we compute the decomposition of  $H^*(P)$  as follows. We first study cohomologies of non-dominant summands (i.e., compute the decomposition of proper subgroups  $Q \subset P$ ). Next compute  $H^*(Y_V)$  for a simple  $Out(P)$ -module  $V$  by using above Lemma 3.3. Then we compute  $H^*(X_{S(P,P,V)})$  from Corollary 3.2 by considering non-dominant summands mainly using the transfer map. Thus we get the decomposition from Theorem 2.2.

#### 4. METACYCLIC GROUPS FOR $p \geq 3$

For  $p \geq 5$ , groups  $P$  with  $rank_p P = 2$  are classified by Blackburn (see Thomas [Th], Dietz-Priddy [Di-Pr]). They are metacyclic groups, groups  $C(r)$  and  $G(r', e)$ . In this section, we consider metacyclic  $p$  groups  $P$  for  $p \geq 3$

$$0 \rightarrow \mathbb{Z}/p^m \rightarrow P \rightarrow \mathbb{Z}/p^n \rightarrow 0. \quad (4.1)$$

These groups are represented as

$$P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a, b] = a^{rp^\ell} \rangle \quad r \neq 0 \bmod(p). \quad (4.2)$$

It is known by ([Hu],[Th]) that  $H^{even}(P; \mathbb{Z})$  is multiplicatively generated by Chern classes of complex representations. Let us write

$$\begin{cases} y = c_1(\rho), & \rho : P \rightarrow P/\langle a \rangle \rightarrow \mathbb{C}^* \\ v = c_{p^m-\ell}(\eta), & \eta = \text{Ind}_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}} \rangle \rightarrow \langle a \rangle \rightarrow \mathbb{C}^* \end{cases}$$

where  $\rho, \xi$  are nonzero linear representations.

By using Quillen's theorem and the fact that  $P$  has just one conjugacy class of maximal abelian  $p$ -subgroups, we can prove

**Theorem 4.1.** (*Theorem 5.45 in [Ya]*) *For any metacyclic  $p$ -group  $P$  with  $p \geq 3$ , we have a ring isomorphism*

$$H^*(P) \cong k[y, v], \quad |v| = 2p^{m-\ell}.$$

For a non split metacyclic groups, it is proved that  $BP$  itself is irreducible [Di]. Hence we consider a split metacyclic group, it is written as

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle \quad (4.3)$$

for  $m > \ell \geq \max(m - n, 1)$ . The outer automorphism is the semidirect product

$$\text{Out}(P) \cong (p - \text{group}) : \mathbb{Z}/(p - 1).$$

The  $p$ -group acts trivially on  $H^*(P)$ , and  $j \in \mathbb{Z}/(p - 1)$  acts as  $a \mapsto a^j$  on  $P$ , and it acts on  $H^*(P)$  as  $j^* : v \mapsto jv$ .

There are  $p - 1$  simple  $\mathbb{Z}/(p - 1)$ -modules  $S_i \cong k\{v^i\}$ . We consider the decomposition by idempotens for  $\text{Out}(P)$ . Let us write  $Y_i = e_{S_i}BP$  and

$$H^*(Y(S_i)) \cong (\dim(S_i))H^*(Y_i) \subset H^*(P)$$

(in the notation  $Y_i$  from Lemma 2.1). Hence we have the decomposition for  $\text{Out}(P)$ -idempotents

$$H^*(Y_i) \cong k[y, V]\{v^i\}, \quad V = v^{p-1}.$$

We assume  $P \neq M(1, 2, 1)$ . Then  $\text{Tr}_H^P(x) = 0$  for  $x \in H^*(H)$  for each proper subgroup  $H$  of  $P$ . By [Di], we have splitting

$$(*) \quad BP \cong \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{i=0}^{p-2} \bar{L}(1, i).$$

Here we write  $X_i = e_{S(P, P, S_i)}BP$  identifying  $S_i$  as the  $A(P, P)$  simple module (but not the simple  $\text{Out}(P)$ -module).

The summand  $\bar{L}(1, i)$  is defined as follows. Recall that  $H^*(\langle b \rangle) \cong k[y]$ . We get  $B\langle b \rangle \cong \bigvee_{i=0}^{p-2} \bar{L}(1, i)$ , with  $H^*(\bar{L}(1, i)) \cong k[Y]\{y^i\}$ . Let  $\Phi \in A(P, P)$  be defined by the map  $\Phi : P \geq P \rightarrow \langle b \rangle \subset P$  which induces

the isomorphisms  $\Phi \cdot H^*(P) \cong k[y] \subset H^*(Y_0)$ . This shows  $X_{S(P, \langle b \rangle, S'_i)} \cong \tilde{L}(1, i)$ , and we have

$$(**) \quad Y_i \cong \begin{cases} X_i & i \neq 0 \\ X_0 \vee \bigvee_{j=0}^{p-2} \bar{L}(1, j) & i = 0. \end{cases}$$

**Theorem 4.2.** *Let  $P = M(\ell, m, n)$  with  $(\ell, m, n) \neq (1, 2, 1)$ . Then we have*

$$H^*(X_i) \cong \begin{cases} k[y, V]\{v^i\} & i \neq 0 \\ k[y, V]\{V\} & i = 0. \end{cases}$$

*Proof.* For  $i \neq 0$ , we have  $H^*(Y_i) \cong H^*(X_i)$ . For  $i = 0$ , we see

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\bigvee_{j=0}^{p-2} L(1, j)) \cong k[y, V] \ominus k[y] \cong k[y, V]\{V\}$$

where  $A \ominus B \cong C$  means  $A \cong B \oplus C$ .  $\square$

For the case  $(\ell, m, n) = (1, 2, 1)$ , see [Hi-Ya3]. In this case we see  $Tr \neq 0$  in  $A(P, P)$ .

## 5. $C(r)$ GROUPS FOR $p \geq 3$

The group  $C(r)$ ,  $r \geq 3$  is the  $p$ -group of order  $p^r$  such that

$$C(r) = \langle a, b, c | a^p = b^p = c^{p^{r-2}} = 1, [a, b] = c^{p^{r-3}} \rangle$$

for  $r \geq 3$ . (In particular,  $C(3) = p_+^{1+2}$ .) Hence we have a central extension

$$0 \rightarrow \mathbb{Z}/p^{r-2} \rightarrow C(r) \rightarrow \mathbb{Z}/p \times \mathbb{Z}/p \rightarrow 0.$$

For each  $r \geq 3$ , the cohomology  $H^*(C(r))$  is isomorphic to  $H^*(C(3))$ . Denote  $C(3) = p_+^{1+2}$  by  $E$ . The cohomology of  $E$  is well known ([Lw], [Le])

$$H^*(E) \cong (k[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus k\{C\}) \otimes k[v]. \quad (5.1)$$

Here  $y_1$  (resp.  $y_2$ ) is the first Chern class  $c_1(e_1)$  (resp.  $c_1(e_2)$ ) for the nonzero linear representation  $e_1 : E \rightarrow \langle a \rangle \rightarrow \mathbb{C}^*$  (resp.  $e_2 : E \rightarrow \langle b \rangle \rightarrow \mathbb{C}^*$ ). The elements  $C$  and  $v$  are also represented by Chern classes

$$c_i(Ind_A^E(e)) = \begin{cases} v & \text{for } i = p \\ C & \text{for } i = p - 1 \end{cases}$$

where  $e : A \rightarrow \langle c \rangle \rightarrow \mathbb{C}^*$  is a non zero linear representation, for any maximal elementary abelian subgroup  $A$ . Hence  $|y_i| = 2, |C| = 2(p-1), |v| = 2p$ . It is known  $Cy_i = y_i^p$ ,  $C^2 = Y_1 + Y_2 - Y_1 Y_1$  where  $Y_i = y_i^{p-1}$  and  $V = v^{p-1}$ .



From the formula (4.1), we get the another expression of  $H^*(E)$  (Proposition 9 in [Gr-Le])

$$H^*(E) \cong k[C, v]\{y_1^i y_2^j \mid 0 \leq i, j \leq p-1, (i, j) \neq (p-1, p-1)\}. \quad (5.2)$$

For  $j = (p-1) + i$  with  $0 \leq i \leq p-2$ . Define  $T(A)^i$  by

$$H^j(E) \supset T(A)^i, \quad T(A)^i = k\{y_1^{p-1} y_2^i, y_1^{p-2} y_2^{i+1}, \dots, y_1^i y_2^{p-1}\}.$$

Then we can identify  $T(A)^i$  as an  $Out(E)$ -module such that  $T(A)^i \cong S(A)^{p-1-i} \otimes det^i[2i]$ . In fact, from (5.2), we also have

**Theorem 5.1.** (*Theorem 4.4 in [Hi-Ya1]*) *Let us write  $\mathbb{CA} = H^*(E)^{Out(E)} \cong k[C, V]$ . Then there is a decomposition of  $Out(E)$ -module such that*

$$H^*(E) \leftrightarrow \mathbb{CA} \otimes \left( \bigoplus_{q=0}^{p-2} \bigoplus_{i=0}^{p-2} (S(A)^i \otimes v^q \oplus T(A)^i \otimes v^q) \right)$$

where  $S(A)^i \otimes v^q \cong S(A)^i \otimes det^q$  and  $T(A)^i \otimes v^q \cong S(A)^{p-1-i} \otimes det^{i+q}[2i]$ .

(I)  $P = C(r)$  for  $r > 3$ .

By Dietz and Priddy, the stable splitting is known. The splitting is given as

$$BP \cong \bigvee_{i,q} (i+1)X_{i,q} \vee \bigvee_q (q+1)L(1, q) \vee pL(1, p-1)$$

where  $0 \leq i \leq p-1$ ,  $0 \leq q \leq p-2$  and  $L(1, p-1) = L(1, 0)$ . Transfers from proper subgroups are always zero when  $r > 3$ .

**Theorem 5.2.** *Let  $P = C(r)$  and  $r \geq 4$ . Then*

$$(i+1)H^*(X_{i,q}) \cong \begin{cases} (i+1)H^*(Y_{i,q}) & \text{if } q \neq 0 \\ \mathbb{CA} \otimes (S(A)^i)\{V\} \oplus T^{p-1-i}(A)v^i & q = 0, i \neq p-1 \\ \mathbb{CA} \otimes S(A)^{p-1}\{V\} & q = 0, i = p-1. \end{cases}$$

(II)  $C(3) = p_+^{1+2}$ .

In this case, the decomposition of cohomology is given in [Hi-Ya1] but it is quite complicated. By Dietz-Priddy, the splitting is given as

$$BP \cong \bigvee_{i,q} (i+1)X_{i,q} \vee \bigvee_q ((p+1)L(2, q) \vee (q+1)L(1, q)) \vee pL(1, p-1)$$

where  $0 \leq i \leq p-1$  and  $0 \leq q \leq p-2$ . The different places from  $r \geq 4$  are the existence of

$$L(2, q) = X_S \quad \text{for } S = S(P, A, S(A)^{p-1} \otimes det^q),$$

which are induced from the transfer (see §9 in [Hi-Ya1] for details). For the cohomology  $H^*(X_{i,q})$  see also [Hi-Ya1].

## 6. $G(r, e)$ FOR $p \geq 5$

Throughout this section, we assume  $p \geq 5$ . The group  $G = G(r, e)$ ,  $r \geq 4$  (and  $e$  is 1 or a quadratic non residue modulo  $p$ ) is defined as

$$\langle a, b, c | a^p = b^p = c^{p^{r-2}} = [b, c] = 1, [a, b^{-1}] = c^{ep^{r-3}}, [a, c] = b \rangle.$$

The subgroup  $\langle a, b, c^p \rangle$  is isomorphic to  $C(r-1)$ . Hence we have the extension

$$1 \rightarrow C(r-1) \rightarrow G(r, e) \rightarrow \mathbb{Z}/p \rightarrow 0.$$

Of course  $E = C(3) \subset C(r-1) \subset G(r, e)$ .

By [Ya], we have an isomorphism

$$H^*(G(r, e)) \cong H^*(E)^{(c)}.$$

The invariant ring  $H^*(C(3))^{(c)}$  is multiplicatively generated by

$$y_1, C, v, y_2^i w \quad \text{where } w = y_2^p - y_1^{p-1} y_2, \quad 0 \leq i \leq p-3$$

since  $c^* : y_2 \mapsto y_2 + y_1$  and  $C^2 = Y_1^2 + y_2^{p-2} w$ . Hence we have

**Lemma 6.1.** *We have isomorphisms*

$$\begin{aligned} (1) \quad H^*(G(r, e)) &\cong (k[y_1] \oplus k[y_2]\{w\} \oplus k\{C\}) \otimes k[v] \\ &\cong \mathbb{C}\mathbb{A} \otimes \bigoplus_{q=0}^{p-2} (k\{1, y_1, \dots, y_1^{p-1}\}\{v^q\} \oplus k\{1, y_2, \dots, y_2^{p-3}\}\{wv^q\}). \end{aligned}$$

**Corollary 6.2.** *We have additively  $H^*(G(r, e)) \cong \bigoplus_{i,q} H^*(Y_{i,q}(E))$ .*

The outer automorphism is  $\text{Out}(P) \cong (p\text{-group}) : (\mathbb{Z}/2 \times \mathbb{Z}/(p-1))$  (see [Di-Pr] for details). Here the action  $i \in \mathbb{Z}/2$  induces  $i : a \mapsto a^{-1}$  and  $k \in \mathbb{Z}/(p-1)$  induces  $k : c \mapsto c^k$ . Hence all simple  $\mathbb{Z}/2 \times \mathbb{Z}/(p-1)$ -modules are represented as  $k\{v^i\}$  and  $k\{y_1 v^i\}$  for  $0 \leq i \leq p-2$ . Using this and Lemma 6.1, we get

**Lemma 6.3.** *Let  $P = G(r, e)$  with  $r \geq 4$ . For  $\text{Out}(P)$ -module decomposition component  $Y_{i,q}(P)$  of  $BP$ , we have additively*

$$H^*(Y_{i,q}(P)) \cong \begin{cases} \bigoplus_{j=\text{even}} H^*(Y_{j,q}(E)) & \text{if } i = 0 \\ \bigoplus_{j=\text{odd}} H^*(Y_{j,q}(E)) & \text{if } i = 1 \end{cases}$$

where  $0 \leq i \leq 1$ ,  $0 \leq j \leq p-1$  and  $0 \leq q \leq p-2$ .

(I)  $G(r, e)$  for  $r > 4$ .

The stable splitting is given by Dietz-Priddy [Di-Pr]

$$BG(r, e) \cong \bigvee_{i,q} X_{i,q}(G(r, e)) \vee \bigvee_q X_{p-1,q}(C(r-1)) \vee \bigvee_q L(1, q)$$

where  $i \in \mathbb{Z}/2$  and  $0 \leq q \leq p-2$ .

**Theorem 6.4.** *For  $r > 4$ , we have*

$$H^*(X_{i,q}(G(r, e))) \cong \begin{cases} H^*((\bigvee_{j=ev}^{p-3} X_{j,0}(C(r-1))) \vee L(1, 0)) & \text{if } i = q = 0 \\ H^*(\bigvee_{j=ev}^{p-3} X_{j,q}(C(r-1))) & \text{if } i = 0, q \neq 0 \\ H^*(\bigvee_{j=odd}^{p-2} X_{j,q}(C(r-1))) & \text{if } i = 1 \end{cases}.$$

(II)  $G(4, e)$

In this case cohomology is the same as (I). However the stable splitting is not same as (I) and it is also given by Dietz and Priddy [Di-Pr]

$$BG(r, e) \cong \bigvee_{i,q} X_{i,q}(G(r, e)) \vee \bigvee_q (X_{p-1,q}(C(r-1)) \vee L(2, q) \vee L(1, q))$$

where  $i \in \mathbb{Z}/2$  and  $0 \leq q \leq p-2$ . The problems are only to see that these  $H^*(L(2, q))$  go to what  $H^*(Y_{i,q'})$ . For details see [Hi-Ya3].

## 7. RELATIONS AMONG $BP$ WITH $\text{rank}_p P = 2$ .

The following lemma is immediate from preceding sections.

**Lemma 7.1.** *Let  $P = C(r)$  (or  $G(r+1, e)$ ) for  $r \geq 3$ . Then for  $0 \leq q \leq p-2$ , non-dominant summands are  $L(1, q)$ ,  $L(2, q)$  (and  $X_{p-1,q}(C(r))$  for  $P = G(r+1, e)$ ).*

For stable homotopy spaces  $X, X'$ , let us write  $X \cong_H X'$  when  $H^*(X) \cong H^*(X')$  as graded modules. Theorem 1.1 in the introduction is a immediate consequence of the above lemma and the following theorem about dominant summands, which follows, for example, from Theorem 6.4 when  $G = G(r, e)$ ,  $r > 4$ .

**Theorem 7.2.** *Let  $P = C(r)$  (or  $G(r+1, e)$ ) for  $r \geq 3$ . Given  $0 \leq i \leq p-1$  (or  $i = 0$  or  $1$ ) and  $0 \leq q \leq p-2$ , there are  $0 \leq a_j, b_k, c \leq 1$  such that we have the isomorphism*

$$X_{i,q}(P) \cong_H \bigvee_{j=0}^{p-1} a_j X_{j,q}(E) \vee \bigvee_{k=0}^{p-2} b_k L(2, k) \vee c L(1, 0)$$

*In particular,  $c = 1$  if and only if  $i = q = 0$  and  $P = G(r+1, e)$ .*

Next, we study split metacyclic groups. For stable spaces  $X = X_{i,j}(C(r))$  or  $X = Y_{i,j}(C(r))$ , let  $SX$  be the virtual object defined by (strictly the module  $H^*(SX)$  is defined)

$$H^*(SX) = H^*(X) \cap \mathbb{CA} \otimes \left( \bigoplus_{q=0}^{p-2} k\{1, y_1, \dots, y_1^{p-2}\} \{v^q\} \right)$$

where we identify it as the submodule of  $\mathbb{C}\mathbb{A} \otimes (\bigoplus_q S(A)^* \{v^q\}) \subset H^*(E)$  in Theorem 4.1. Then we see

$$H^*(S(BE)) \cong \mathbb{C}\mathbb{A} \otimes \left( \bigoplus_q (k\{1, y_1, \dots, y_1^{p-2}\} \{v^q\}) \right) \cong k[y_1, v]$$

identifying  $C = Y = y_1^{p-1}$  as graded modules.

Recall that for the split metacyclic group  $M = M(\ell, m, n)$ , we have  $H^*(M) \cong k[y, v]$  with  $|v| = 2p^{m-\ell}$  from Theorem 4.1. In particular, when  $m - \ell = 1$ , we see  $H^*(M) \cong H^*(S(BE))$ . The results in §5 imply the following theorem

**Theorem 7.3.** *Let  $M = M(m - 1, m, n)$ . Then we have*

$$H^*(X_q(M)) \cong \begin{cases} \bigoplus_{j=0}^{p-2} H^*(SX_{j,q}(C(r))), & \text{for } r > 3, \quad \text{if } (m, n) \neq (2, 1) \\ \bigoplus_{j=0}^{p-2} H^*(SX_{j,q}(E)) & \text{if } (m, n) = (2, 1). \end{cases}$$

At last in this section, we consider the cases  $m - \ell > 1$ . From the results in §5, it is almost immediate

**Proposition 7.4.** *Let  $m - \ell > 1$ . Then we have*

$$H^*(X_i(M(\ell, m, n))) \cong H^*(X_i(M(m - 1, m, n))) \cap k[y, v^{p^{m-\ell-1}}].$$

From these results, we get

**Theorem 7.5.** *For  $p \geq 5$ , let  $P$  be a non-abelian  $p$ -group of  $\text{rank}_p P = 2$ . Then there is a submodule  $HS(P) \subset H^*(E)$  such that for each primitive idempotent  $e$  in  $A(P, P)$ , there is an idempotent  $f \in A(E, E)$  such that  $eH^*(P) \cong HS(P) \cap fH^*(E)$ . When  $P$  is not metacyclic, we can take  $HS(P) = H^*(E)$ .*

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